View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Calhoun, Institutional Archive of the Naval Postgraduate School

Calhoun

Institutional Archive of the Naval Postgraduate School

Calhoun: The NPS Institutional Archive

Faculty and Researcher Publications | Faculty and Researcher Publications Collection

1995

A Velocity Algorithm for the Implementation of Gain-Scheduled Controllers

Kaminer, Isaac

I. Kaminer, A.M. Pascoal, P.P. Khargonekar, and E. Coleman, "A Velocity Algorithm for the Implementation of Gain-Scheduled Controllers'', Automatica, Vol. 31, pp. 1185--1191, 1995.

DUDLEY KNOX LIBRARY

Calhoun is a project of the Dudley Knox Library at NPS, furthering the precepts and goals of open government and government transparency. All information contained herein has been approved for release by the NPS Public Affairs Officer.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

http://www.nps.edu/library

Pergamon

0005-1098(95)00026-7

Automatica, Vol. 31, No. 8, pp. 1185-1191, 1995
Copyright © 1995 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0005-1098/95 $9.50 + 0.00

Brief Paper

A Velocity Algorithm for the Implementation of Gain-scheduled Controllers*

ISAAC KAMINER,† ANTONIO M. PASCOAL,‡ PRAMOD P. KHARGONEKAR§ and EDWARD E. COLEMAN||

Key Words—Nonlinear control; gain scheduling; robust control; controller implementation.

Abstract—A new method is proposed to implement gain-scheduled controllers for nonlinear plants. Given a family of linear feedback controllers designed for linearizations of a nonlinear plant about constant operating points, a nonlinear gain-scheduled controller is derived that preserves the input–output properties of the linear closed loop systems locally, about each equilibrium point. The key procedures in the proposed method are to provide integral action at the inputs to the plant and differentiate some of the measured outputs before they are fed back to the scheduled controller. For a fairly general class of systems, the nonlinear gain-scheduled controllers are easy to obtain, and their structure is similar to that of the original linear controllers.

1. *Introduction*

This paper addresses the problem of implementing gain scheduled controllers for nonlinear plants. Traditionally, the development of such controllers involves the following steps.

Step 1. Linearize the plant about a finite number of representative operating points.

Step 2. Design linear controllers for the plant linearizations at each operating point.

Step 3. Interpolate the parameters of the linear controllers of Step 2 to achieve adequate performance of the linearized closed loop system at all points where the plant is expected to operate—the resulting family of linear controllers is called a gain-scheduled controller.

Step 4. Implement the gain-scheduled controller on the nonlinear plant.

The resulting gain-scheduled controller is nonlinear, its parameters evolving as functions of the plant states, inputs, outputs and exogenous parameters, or any combination thereof. As an illustrative example, consider the design of a controller for an airplane. First, the nonlinear equations of motion are linearized about selected operating points that capture the key modes of operation throughout the flight envelope (the set of conditions under which the airplane is expected to fly). Linear controllers are then designed to achieve desired stability and performance requirements for the linearizations of the plant about the selected operating points. Since these requirements must be satisfied throughout the flight envelope, the parameters of the controllers are then interpolated as a function of a gain scheduling variable—typically, this variable can be dynamic pressure, Mach number, altitude, angle of attack or a combination of the above. Finally, the scheduled controller is adequately modified and implemented on the nonlinear plant.

This technique has proven successful in many engineering applications. See for example Åström and Wittenmark (1988) and Stein *et al.* (1977) for interesting applications in the areas of flight control, ship steering, combustion control and control of the air/fuel ratio in car engines. However, designing a gain-scheduled controller remains largely an *ad hoc* procedure that requires extensive computer simulations. This is due to the lack of powerful analysis tools to assess the stability and performance of the resulting nonlinear time-varying feedback systems.

In recent years considerable progress has been made in the theory of nonlinear gain-scheduled control systems that arise from scheduling on a reference trajectory or on the plant outputs (see Shamma, 1988; Shamma and Athans 1990; and references therein). The main results are a set of conditions, albeit conservative, guaranteeing that the properties of robust stability and performance of the linear frozen-time feedback systems obtained at each operating point carry over to the global gain-scheduled system. In practice, these conditions formalize rules of thumb for the design of gain-scheduled systems.

Assuming those conditions hold, it is then natural that the development of gain-scheduled controllers proceed as follows: (i) design a family of linear controllers to achieve satisfactory closed-loop dynamic behavior for the linear designs at each operating point—this is within the scope of linear control theory; (ii) gain-schedule the resulting linear controllers so that the following fundamental property holds.

Linearization property. At any given operating point, the linearization of the feedback system consisting of the gain-scheduled controller and the nonlinear plant (frozen-time system) exhibits the same internal and input–output properties as the feedback interconnection of the linearized plant and the corresponding linear controller. (See Step 2 above.)

Surprisingly, this property is not satisfied in some gain-scheduled systems described in the literature. See for example the comments in Shamma and Athans (1990) and on p. 904 of Shamma (1988). Whereas most of the research effort has concentrated on the design and analysis of the gain-scheduled controllers described in Steps 2 and 3 above, the issue of properly implementing such controllers (Step 4) so that the linearization property holds has largely been ignored. In fact, many schemes resort to direct implementation of the linear gain-scheduled controllers obtained in Step 3 on the nonlinear plant. This may result in a loss of performance or even instability of the feedback system linearized about one of the operating points. Clearly, in this

* Received 24 August 1993; revised 18 May 1994; received in final form 3 January 1995. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Hassan Khalil under the direction of Editor Tamer Başar. Corresponding author Professor Pramod P. Khargonekar. Tel. +1 313 764 4328; Fax +1 313 763 8041; E-mail pramod@eecs.umich.edu.

† Department of Aeronautical and Astronautical Engineering, Naval Postgraduate School, Monterey, CA 93943, U.S.A.

‡ Institute for Systems and Robotics (ISR) and Department of Electrical Engineering, Instituto Superior Tecnico, Av. Rovisco Pais, 1096 Lisbon Codex, Portugal.

§ Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, MI 48109-2122, U.S.A.

|| Boeing Company, Seattle, WA, U.S.A.

case not even the properties guaranteed in Step 3 are recovered locally.

The importance of this issue has been acknowledged in the control literature in a series of papers by Rugh and his co-workers devoted to extended linearization techniques (Baumann and Rugh, 1986; Wang and Rugh, 1987; Bauman, 1988; Lawrence and Rugh, 1993; Rugh, 1990, 1991). For single-input multi-output nonlinear systems, a technique has been reported by Baumann and Rugh (1986) for the design of state feedback and observer/state feedback based controllers such that the eigenvalues of the family of linearized closed-loop systems are invariant with respect to the operating point. The extension of these results to the multi-input multi-output case can be found in Baumann and Rugh (1987) and Baumann (1988), where the authors show that nonlinear state feedback and observer/state feedback control laws can be derived as to place the eigenvalues of the family of closed-loop linearizations at specified locations, which may be a function of the closed-loop operating point. In a very general set-up, the procedure requires the solution of partial differential equations that involve nontrivial integrability conditions. A related paper by Wang and Rugh (1987) establishes necessary and sufficient conditions on general linear control laws that can arise when a nonlinear dynamic feedback control law for a given nonlinear system is linearized about a family of operating points. The sufficiency proof is constructive in the sense that it provides the recipe for building the corresponding gain-scheduled controller. In the general case, however, the procedure also requires the solution of partial differential equations of considerable difficulty.

In this paper we derive a new, simple method for the implementation of gain-scheduled controllers for nonlinear plants, so that the linearization property holds. The method can be applied to a fairly general class of control structures that are usually referred to as tracking controllers. Given a family of linear feedback controllers designed for linearizations of the nonlinear plant about constant operating points, a nonlinear gain-scheduled controller is derived that preserves the internal and input–output properties of the linear closed-loop systems locally, about each equilibrium point. The method is based on the observation that linear controllers obtained in Step 2 above are designed to operate on the perturbations of the plant's inputs and outputs about the equilibrium points. Proper blending of the different controllers requires that they have access to such perturbations, locally. This is achieved by differentiating some of the measured outputs before they are fed back to the gain-scheduled controller. In order to preserve the input–output behaviour of the feedback system, integral action is provided at the input to the plant. The resulting nonlinear gain-scheduled controller is easy to obtain, and its structure is similar to that of the original linear controller. Furthermore, despite the use of differentiators, this scheme does not introduce additional noise amplification at the relevant inputs and outputs of the linearized feedback system, since all closed-loop transfer functions are preserved. The issue of noise amplification inside the controller and how it impacts on the behavior of the nonlinear feedback system is not addressed in this paper. On the positive side, at the level of local linear analysis, since all closed-loop transfer functions are preserved, it follows that no extra noise amplification is introduced by our scheme.

It is important to emphasize that a potential weak point of this method, namely the requirement that some of the outputs be differentiated, can be dealt with in practice by using suitable approximations: on an analog computer, differentiation can be replaced by a causal operation while having the resulting system satisfy the linearization property asymptotically; in the case of digital computers differentiation is replaced by taking the difference between present and previous values of the measured outputs. See Kaminer *et al.* (1994) for complete details.

The method proposed here bears a close connection to so-called velocity algorithms for the implementation of PID controllers in process control (Isermann, 1981). Since in practice derivatives are often implemented numerically using finite differences, the method will be referred to as the $\mathscr{D}$ method for gain-scheduled control. The $\mathscr{D}$ method was first conceived by E. Coleman, C. Thompson and P. Salo (see Salo, 1987), while working for the Boeing Company. Since then, it has been successfully applied to a number of flight control problems. The reader is referred to Kaminer *et al.* (1990) for an interesting application to the design of a lateral autopilot for an airplane. See also Kaminer *et al.* (1991) for the implementation details of a trajectory-tracking controller for an underwater vehicle.

It should be stressed that this paper does not include a discussion of the nonlinear properties of the proposed implementation, namely the extent of asymptotic stability of the nonlinear gain-scheduled system. This important issue warrants further research.

The paper is organized as follows. Section 2 summarizes the basic notation, and Section 3 provides the problem formulation. Section 4 introduces the $\mathscr{D}$ method for the implementation of continuous-time gain-scheduled tracking controllers. Finally, Section 5 describes an approximation to the $\mathscr{D}$ method to guarantee the causality of the controller proposed. An early conference version of this paper appeared as Kaminer *et al.* (1993).

2. *Notation*

The following notation will be used. Given a function $f(x, u): \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ of class C^1,

$$\frac{\partial}{\partial x} f(x_0, u_0) \in \mathbb{R}^{p\times m}$$

denotes the derivative of f with respect to x evaluated at (x_0, u_0). Similarly,

$$\frac{\partial}{\partial u} f(x_0, u_0) \in \mathbb{R}^{p\times n}$$

denotes the derivative of f with respect to u evaluated at (x_0, u_0). We shall deal with dynamical systems described by equation of the type

$$\dot{x} = f(x, u), \tag{1}$$

$$y = h(x, u), \tag{2}$$

where $f(x, u): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h(x, u): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are both of class C^1. The vector $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium point of (1), (2) if $f(x_0, u_0) = 0$. The linearization of (1), (2) at (x_0, u_0) is the system defined by

$$\dot{\xi} = A(x_0, u_0)\xi + B(x_0, u_0)\eta, \tag{3}$$

$$\theta = C(x_0, u_0)\xi + D(x_0, u_0)\eta, \tag{4}$$

where

$$A(x_0, u_0) := \frac{\partial}{\partial x} f(x_0, u_0),$$

$$B(x_0, u_0) := \frac{\partial}{\partial u} f(x_0, u_0),$$

$$C(x_0, u_0) := \frac{\partial}{\partial x} h(x_0, u_0),$$

$$D(x_0, u_0) := \frac{\partial}{\partial u} h(x_0, u_0),$$

and ξ, η and θ correspond to small perturbations of x, u and y about x_0, u_0 and $y_0 = h(x_0, u_0)$.

3. *Problem formulation*

Consider the feedback system shown in Fig. 1. The nonlinear system $\mathscr{G}$ consists of a dynamical model of the physical plant to be controlled, together with appended dynamics that shape the exogenous signals w, the control inputs u and the internal input and output variables that contribute to the generalized error signals z. The controller

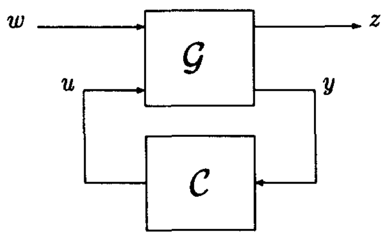


Fig. 1. Feedback interconnection of nonlinear plant $\mathcal{G}$ and controller $\mathcal{C}$.

$\mathcal{C}$ operates on the measured output variables y to produce the control inputs u.

In this paper we consider general tracking control structures whereby some of the plant outputs are required to track (step or slowly varying) reference commands. Therefore, the vector w is decomposed as $w = [d' \; w_m' \; r']'$, where d is the vector of exogenous signals that are not accessible for measurement (e.g. external disturbances and sensor noise), w_m is the vector of measurable exogenous signals (e.g. air speed in an airplane or water speed in an underwater vehicle) and r is the vector of external reference signals to be tracked. Furthermore, $y = [y_1' \; y_2']'$, where y_2 is the vector of output signals that must track the reference commands r and y_1 consists of an extra set of measurable output signals that will be used for feedback. We allow y_1 to include some or all of the components of y_2. This will simplify the exposition, as shown in the sequel. With this notation, the generalized plant $\mathcal{G}$ can be described as

$$\mathcal{G} := \begin{cases} \dot{x} = f(x, w, u), \\ z = g(x, w, u), \\ y_1 = h_1(x, w), \\ y_2 = h_2(x, w), \end{cases} \tag{5}$$

where f, g, h_1 and h_2 are functions of class C^1, $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^l$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^q$ and $y \in \mathbb{R}^p$.

3.1. *Equilibrium points.* We assume that the plant $\mathcal{G}$ has a family of (equilibrium) operating points

$$\mathscr{E} := \left\{ (x_0, w_0, u_0) : \begin{bmatrix} f(x_0, w_0, u_0) = 0 \\ h_2(x_0, w_0) = r_0 \end{bmatrix}, (x_0, w_0, u_0) \in \Omega \right\},$$

where $w_0 = [d_0' \; w_{m_0}' \; r_0']' \in \mathbb{R}^l$ and Ω is an open subset of $\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$. We further assume that the set $\mathscr{E}$ can be parameterized by a vector α_0 in some open subset $\mathscr{A}$ of $\mathbb{R}^s$; that is,

$$\mathscr{E} = \mathscr{E}(\alpha_0) := \{(x_0, w_0, u_0) := (x_0, w_0, u_0) = g(\alpha_0), \alpha_0 \in \mathscr{A}\}$$

where $g: \mathscr{A} \to \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ is continuously differentiable and one-to-one. We remark that the definitions of Ω and α_0 are problem-dependent and are largely dictated by experience. Moreover, α_0 is often determined indirectly by measuring the equilibrium values of some of the inputs and outputs of the plant. Let $\mathscr{S} := \{(w_0, u_0, y_0) : y_0 = h(x_0, w_0), (x_0, w_0, u_0) \in \mathscr{E}\}$. We assume that $\alpha_0 = v(w_0, u_0, y_0)$, where $v: \mathscr{S} \to \mathbb{R}^s$ is a C^1 function. By applying the function v to the measured values of w, u and y, we obtain the variable

$$\alpha := v(w, u, y) \in \mathbb{R}^s, \tag{6}$$

which is usually referred to as the scheduling variable (Wang and Rugh, 1987). We set $d_0(\alpha_0) = 0$ for all $\alpha_0 \in \mathscr{A}$, since the exogenous inputs d are not known in advance and, more importantly, because we are usually interested in the response of the overall feedback system to small perturbations in d about its zero nominal value. Note that the scheduling variable $\alpha := v(w, u, y)$ may depend explicitly on the reference commands r and on the exogenous measurable variables w_m.

3.2. *Linearized plant models.* Given $(x_0, w_0, u_0) \in \mathscr{E}$, set

$$z_0 := g(x_0, w_0, u_0),$$
$$y_{1_0} := h_1(x_0, w_0),$$
$$y_{2_0} := h_2(x_0, w_0),$$

and let ξ, v, η, ζ, θ_1 and θ_2 correspond to small perturbations of x, w, u, z, y_1 and y_2 about $x_0, w_0, u_0, z_0, y_{1_0}$ and y_{2_0} respectively. The family of linear models $\mathcal{G}_\ell$ associated with the plant $\mathcal{G}$ and the set $\mathscr{E}$ is defined as $\mathcal{G}_\ell := \{\mathcal{G}_\ell(x_0, w_0, u_0) : (x_0, w_0, u_0) \in \mathscr{E}\}$, where

$$\mathcal{G}_\ell(x_0, w_0, u_0) := \begin{cases} \dot{\xi} = A(x_0, w_0, u_0)\xi + B_1(x_0, w_0, u_0)v + B_2(x_0, w_0, u_0)\eta, \\ \zeta = C_0(x_0, w_0, u_0)\xi + D_{01}(x_0, w_0, u_0)v + D_{02}(x_0, w_0, u_0)\eta, \\ \theta_1 = C_1(x_0, w_0, u_0)\xi + D_1(x_0, w_0, u_0)v, \\ \theta_2 = C_2(x_0, w_0, u_0)\xi + D_2(x_0, w_0, u_0)v \end{cases} \tag{7}$$

is the linearization of $\mathcal{G}$ at (x_0, w_0, u_0). For notational simplicity, $\mathcal{G}_\ell(x_0, w_0, u_0)$ will often be written simply as $\mathcal{G}_\ell(\alpha_0)$, since it is assumed that the equilibrium points are parameterized by α_0.

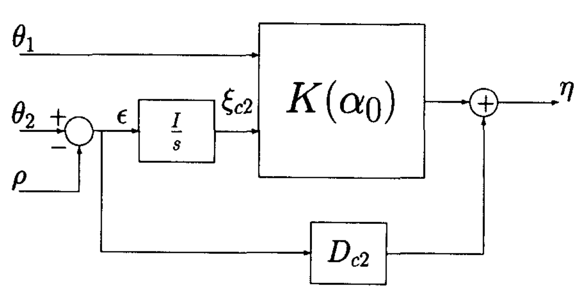
3.3. *Family of linear controllers.* As discussed in Section 1, a common approach to the design of a gain-scheduled controller for $\mathcal{G}$ requires designing a family of linear controllers for a finite number of plants in $\mathcal{G}_\ell$, and then interpolating between controllers as to achieve adequate performance for all the linearized plants in $\mathcal{G}_\ell$. During real-time operation, the controller parameters are updated as functions of the gain-scheduling variable α. In this paper we restrict ourselves to the idealized case where the description of each controller for each plant in $\mathcal{G}_\ell$ is available (Rugh, 1990). Therefore we assume that the first design step produces the set $\mathcal{C}_\ell := \{\mathcal{C}_\ell(\alpha_0) : \alpha_0 \in \mathscr{A}\}$, where $\mathscr{A}$ is the parametrizing set referred to above and $\mathcal{C}_\ell(\alpha_0)$ denotes the controller for the plant $\mathcal{G}_\ell(x_0, w_0, u_0)$. Let $e = y_2 - r$ denote the vector of tracking errors, and let ρ and ϵ correspond to small perturbations of r and e about the equilibrium points r_0 and $e_0 = y_{2_0} - r_0$ respectively. The linear (tracking) controllers $\mathcal{C}_\ell(\alpha_0)$ considered in this paper are described by (see Fig. 2)

$$\mathcal{C}_\ell(\alpha_0) := \begin{cases} \dot{\xi}_{c1} = A_{c1}(\alpha_0)\xi_{c1} + B_{c1}(\alpha_0)\theta_1 + B_{c2}(\alpha_0)\xi_{c2}, \\ \dot{\xi}_{c2} := \epsilon = \theta_2 - \rho, \\ \eta = C_{c1}(\alpha_0)\xi_{c1} + C_{c2}(\alpha_0)\xi_{c2} + D_{c1}(\alpha_0)\theta_1 + D_{c2}(\alpha_0)\epsilon, \end{cases} \tag{8}$$

where $\xi_{c1} \in \mathbb{R}^r$, $\xi_{c2} \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^m$, and the matrices are of compatible dimensions. We further assume that the parameters of the controller are C^1 functions of α_0.

The structure of the linear controller $\mathcal{C}_\ell(\alpha_0)$ has two important features.

(i) Suppose the closed-loop system consisting of (7) and (8) is asymptotically stable. Then the controller $\mathcal{C}_\ell(\alpha_0)$ will ensure zero steady-state error ϵ to a step input in ρ. This is achieved by integrating the error $\epsilon = \theta_2 - \rho$. This structure is typical of tracking controllers, since they are designed to drive errors between step changes in reference commands and the corresponding plant outputs to zero in steady state. Note that the block $K(\alpha_0)$ (see Fig. 2) may itself contain additional integrators.



$$K(\alpha_0) = \left[\begin{array}{c|cc} A_{c1}(\alpha_0) & B_{c1}(\alpha_0) & B_{c2}(\alpha_0) \\ \hline C_{c1}(\alpha_0) & D_{c1}(\alpha_0) & C_{c2}(\alpha_0) \end{array} \right]$$

Fig. 2. Linear controller $\mathcal{C}_\ell(\alpha_0)$.

(ii) A subset θ_1 of the measured outputs of $\mathscr{G}_\ell(\alpha_0)$ is fed back directly to the controller. This controller structure includes controllers with inner/outer loops that are used extensively in aerospace applications.

3.4. *Problem formulation.* The problem that we address in this paper is to find a gain-scheduled (possibly nonlinear) controller

$$\mathscr{C}(\alpha) := \begin{cases} \dot{x}_c = f_c(x_c, y, \alpha), \\ u = h_c(x_c, y, \alpha), \\ \alpha = v(w, u, y) \end{cases} \quad (9)$$

for the nonlinear plant $\mathscr{G}$ so that the linearization property given in Section 1 is satisfied at each equilibrium point determined by $\alpha_0 \in \mathscr{A}$. Let

$$\mathscr{T}(\mathscr{G}_\ell(\alpha_0), \mathscr{C}_\ell(\alpha_0)): v \to \zeta$$

be the closed-loop linear system that results from connecting $\mathscr{C}_\ell(\alpha_0)$ to $\mathscr{G}_\ell(\alpha_0)$, and denote by $T(\mathscr{G}_\ell(\alpha_0), \mathscr{C}_\ell(\alpha_0))(s)$ the corresponding matrix transfer function. Let

$$\mathscr{T}(\mathscr{G}, \mathscr{C}): w \to z$$

be the nonlinear closed loop system that consists of $\mathscr{C}$ and $\mathscr{G}$, and let

$$\mathscr{T}_\ell(\mathscr{G}, \mathscr{C})(\alpha_0)$$

denote its linearization at α_0. Here we are assuming that the equilibrium points of the controller are also parameterized by $\alpha_0 \in \mathscr{A}$ and satisfy $f_c(x_c(\alpha_0), y_0(\alpha_0), \alpha_0) = 0$ and $u_0(\alpha_0) = h_c(x_c(\alpha_0), y_0(\alpha_0), \alpha_0)$. With this notation, the controller implementation problem considered in this paper can be stated as follows.

Controller implementation problem. Find a gain-scheduled controller $\mathscr{C}(\alpha)$ such that for each equilibrium point of $\mathscr{G}$ in $\mathscr{E}(\alpha_0)$ the following properties hold:

(i) the feedback systems $\mathscr{T}_\ell(\mathscr{G}, \mathscr{C})(\alpha_0)$ and $\mathscr{T}(\mathscr{G}_\ell(\alpha_0), \mathscr{C}_\ell(\alpha_0))$ have the same closed-loop eigenvalues;
(ii) the closed-loop transfer functions $T_\ell(\mathscr{G}, \mathscr{C})(\alpha_0)(s)$ and $T(\mathscr{G}_\ell(\alpha_0), \mathscr{C}_\ell(\alpha_0))(s)$ are equal.

4. *The $\mathscr{D}$-methodology for the implementation of gain-scheduled tracking controllers*

This section provides a complete solution to the controller implementation problem formulated in Section 3. For the sake of clarity, we first illustrate the key features of the proposed solution with a simple example.

4.1. *Motivating example.* This example is based on the simplified one-dimensional motion dynamics of an underwater vehicle (Slotine and Li, 1991). The vehicle is driven by the (propeller-generated) force u, moves with forward velocity x with respect to the water, and is subjected to a drag force proportional to $x\,|x|$. The objective is to design a gain-scheduled controller to steer the vehicle along a desired velocity profile.

In this case the plant $\mathscr{G}$ is described by

$$\mathscr{G} := \begin{cases} m\dot{x} = -cx\,|x| + u, \\ y = x + d, \end{cases} \quad (10)$$

where d is the noise that corrupts the velocity measurement, and m and c denote the total mass (including so-called added mass) of the vehicle and the drag coefficient respectively. For simplicity, we assume that $c = m = 1$. Consider the set of equilibrium points

$$\mathscr{E} := \{(x_0, u_0): u_0 = -cx_0\,|x_0|,\ x_0 \in (x_{0\min}, x_{0\max}),\ x_{0\max} > x_{0\min} > 0\},$$

and let the scheduling variable α be the measured velocity of the vehicle, that is, $\alpha = y$. The linearization of $\mathscr{G}$ at each equilibrium point $\alpha_0 = y_0 = x_0$ is

$$\mathscr{G}_\ell(x_0) := \begin{cases} \dot{\xi} = -2x_0\xi + \eta, \\ \theta = \xi + \delta. \end{cases} \quad (11)$$

Consider a traditional proportional-plus-integral controller for this plant, described by

$$\mathscr{C}_\ell(x_0) := \begin{cases} \dot{\xi}_c = \xi - \rho, \\ \eta = -\xi_c - (\beta - 2x_0)\theta, \quad \beta > 0. \end{cases} \quad (12)$$

Simple algebra shows that the characteristic polynomial of the feedback interconnection of the linearized plant (11) and the controller (12) is $\lambda^2 + \beta\lambda + 1$, and that the closed-loop transfer functions $\hat{\xi}(s)/\hat{\rho}(s)$ and $\hat{\xi}(s)/\hat{\delta}(s)$ equal $1/(s^2 + \beta s + 1)$ and $(s(2x_0 - \beta) - 1)/(s^2 + \beta s + 1)$ respectively. Note that in this particular example the closed-loop eigenvalues are independent of the equilibrium point. However, this is not essential to the proposed solution. Since the controller parameters are given explicitly as functions of the equilibrium point, there is no need to interpolate between them.

Consider the (rather naive) gain-scheduled controller

$$\mathscr{C}_u(y) := \begin{cases} \dot{x}_c = y - r, \\ u = -x_c - (\beta - 2y)y \end{cases} \quad (13)$$

that is obtained by simply mimicking the structure of the linear controller. The linearization of the feedback system that consists of the plant $\mathscr{G}$, (10), and the controller $\mathscr{C}_u$, (13), has characteristic polynomial $s^2 + (\beta - 2x_0)s + 1$. If $x_0 \le \frac{1}{2}\beta$, the linearized system becomes unstable. This happens because the scheduling variable $\alpha = y$ introduces additional dynamics that show up in the linearization procedure. Notice how the linearization of the term $(\beta - 2y)y$ about the operating point x_0 gives $(\beta - 4x_0)\theta$ and not $(\beta - 2x_0)\theta$ as desired (see (12)). Clearly, this implementation leads to the wrong results.

Consider now the gain-scheduled controller $\mathscr{C}(y)$ formed by moving the integrator of $\mathscr{C}_u(y)$ in front of the plant $\mathscr{G}$:

$$\mathscr{C}(y) := \begin{cases} \dot{x}_c = -y + r - (\beta - 2y)\dot{y}, \\ u = x_c. \end{cases} \quad (14)$$

The feedback system consisting of the plant $\mathscr{G}$, (10), and controller $\mathscr{C}$, (14), can be linearized about the equilibrium point $(x_0, r_0 = x_0 = y_0, d_0 = 0, x_{c0} = x_0^2)$, leading to the closed-loop equations

$$\dot{\xi} = -2x_0\xi + \xi_c, \quad (15)$$

$$\dot{\xi}_c = (-1 + 2x_0\beta - 4x_0^2)\xi + (2x_0 - \beta)\xi_c + \rho - \delta + (2x_0 - \beta)\dot{\delta}. \quad (16)$$

It is trivial to check that the eigenvalues of the closed-loop system (15), (16) are the roots of the polynomial $s^2 + \beta s + 1$. Moreover, the closed-loop transfer functions from ρ and δ to ξ are $1/(s^2 + \beta s + 1)$ and $(s(2x_0 - \beta) - 1)/(s^2 + \beta s + 1)$ respectively. With the implementation (14), the closed-loop eigenvalues and the closed-loop transfer functions of the linear designs are preserved at each operating point. Clearly, this implementation avoids the problems encountered by the previous one. In the next subsection we present the solution to the controller implementation problem that was used to obtain the realization (14).

4.2. *Main result.* Given the set $\mathscr{C}_\ell$ of linear controllers for the family $\mathscr{G}_\ell$ of linearized plant models, we propose the following structure for the gain-scheduled controller $\mathscr{C}(\alpha)$ (see Fig. 3);

$$\mathscr{C}(\alpha) := \begin{cases} \dot{x}_{c1} = A_{c1}(\alpha)x_{c1} + B_{c1}(\alpha)\dot{y}_1 + B_{c2}(\alpha)e, \\ \dot{x}_{c2} = C_{c1}(\alpha)x_{c1} + D_{c1}(\alpha)\dot{y}_1 + C_{c2}(\alpha)e, \\ e = y_2 - r, \\ u = x_{c2} + D_{c2}(\alpha)e, \\ \alpha = v(w, u, y). \end{cases} \quad (17)$$

Equations (17) will be referred to as the $\mathscr{D}$-controller implementation methodology, since they require that the output signal y_1 be differentiated. Note in Figs 2 and 3 that the structure of the gain-scheduled controller is easily obtained from that of the linear controllers.

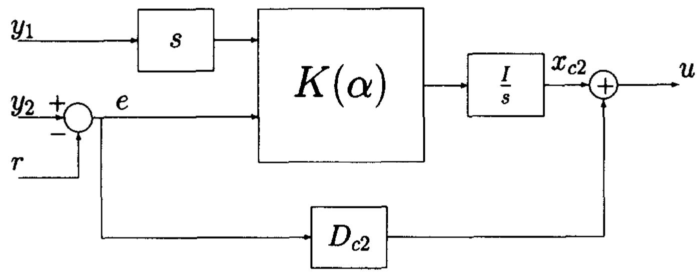


Fig. 3. $\mathscr{D}$ implementation of $\mathscr{C}_\ell(\alpha)$.

We make the following assumptions:

(A1) $\dim(x_{c2}) = \dim(u) = \dim(y_2)$;

(A2) the matrix

$$\begin{bmatrix} sI - A_{c1}(\alpha_0) & B_{c2}(\alpha_0) \\ -C_{c1}(\alpha_0) & C_{c2}(\alpha_0) \end{bmatrix}$$

has full rank at $s = 0$ for each $\alpha_0 \in \mathscr{A}$;

(A3) $w \in C^1[0, \infty)$ or $y_1 = y_1(x)$; that is, y_1 is only a function of x.

Assumption (A1) implies that the number of integrators is equal to the number of control inputs. This is necessary if the controller is to provide independent control of the measured outputs y_2 using the control inputs u. Assumption (A2) implies that the realization $(A_{c1}, B_{c2}, C_{c1}, C_{c2})$ has no transmission zeroes at the orign. Assumption (A3) is sufficient to ensure that $\dot{y}_1$ exists and is continuous.

The main result of this section is as follows.

Theorem 4.1. Suppose Assumptions (A1)–(A3) hold. Then the gain-scheduled controller $\mathscr{C}(\alpha)$ given by (17) solves the controller implementation problem, i.e., for each equilibrium point of $\mathscr{G}$ in $\mathscr{E}(\alpha_0)$, the following properties hold:

(i) the feedback systems $\mathscr{T}_\ell(\mathscr{G}, \mathscr{C})(\alpha_0)$ and $\mathscr{T}(\mathscr{G}_\ell(\alpha_0), \mathscr{C}_\ell(\alpha_0))$ have the same closed-loop eigenvalues;

(ii) the closed-loop matrix transfer functions $T_\ell(\mathscr{G}, \mathscr{C})(\alpha_0)(s)$ and $T(\mathscr{G}_\ell(\alpha_0), \mathscr{C}_\ell(\alpha_0))(s)$ are equal.

Proof. We set the controller throughput matrices D_{c1} and D_{c2} equal to zero. This does not change the results of the theorem, and considerably simplifies the algebra. Let $\alpha_0 \in \mathscr{A}$ be given, and consider the feedback interconnection of the linearized plant $\mathscr{G}_\ell(\alpha_0)$ and the corresponding linear controller $\mathscr{C}_\ell(\alpha_0)$. The state matrix F of the feedback system can be written as

$$F := \begin{bmatrix} A & B_2C_{c1} & B_2C_{c2} \\ B_{c1}C_1 & A_{c1} & B_{c2} \\ C_2 & 0 & 0 \end{bmatrix}, \tag{18}$$

where for notational simplicity we have omitted writing the explicit dependence of the matrix on α_0. Next we linearize the feedback interconnection of the plant $\mathscr{G}$ and the controller $\mathscr{C}(\alpha)$, shown in Fig. 3, at the equilibrium point $(x_0, w_0, x_{c1_0}, x_{c2_0})$ determined by α_0. First, we proceed to determine the states of the controller corresponding to this equilibrium point. Consider the set of algebraic equations

$$A_{c1}(\alpha_0)x_{c1_0} + B_{c1}(\alpha_0)\dot{y}_{1_0} + B_{c2}(\alpha_2)(y_{2_0} - r_0) = 0, \tag{19}$$

$$C_{c1}(\alpha_0)x_{c1_0} + C_{c2}(\alpha_0)(y_{2_0} - r_0) = 0, \tag{20}$$

$$u_0 = x_{c2_0}. \tag{21}$$

Since y_{1_0} is constant, it follows that $\dot{y}_{1_0} = 0$. Therefore (19) and (20) can be written in matrix form as

$$\begin{bmatrix} A_{c1}(\alpha_0) & B_{c2}(\alpha_0) \\ C_{c1}(\alpha_0) & C_{c2}(\alpha_0) \end{bmatrix} \begin{bmatrix} x_{c1_0} \\ y_{2_0} - r_0 \end{bmatrix} = 0. \tag{22}$$

Assumptions (A1)–(A2) imply that the matrix

$$\begin{bmatrix} A_{c1}(\alpha_0) & B_{c2}(\alpha_0) \\ C_{c1}(\alpha_0) & C_{c2}(\alpha_0) \end{bmatrix} \tag{23}$$

is square and invertible for each $\alpha_0 \in \mathscr{A}$. It then follows from the assumptions and (21) and (22) that

$$y_{2_0} - r_0 = 0, \quad x_{c1_0} = 0, \quad c_{c2_0} = u_0.$$

In order to compute the linearization of the feedback interconnection of $\mathscr{G}$ and $\mathscr{C}(\alpha)$, we must first obtain the linearizations of (5) and (17) about the operating points (x_0, w_0, u_0) and $(x_{c1_0} = 0, x_{c2_0} = u_0, \dot{y}_{1_0} = 0, e_0 = y_{2_0} - r_0 = 0)$ respectively, determined by α_0. The linearization of the plant $\mathscr{G}$ is given by (7). The linearization of the controller $\mathscr{C}(\alpha)$ can be written as

$$\begin{aligned}
\dot{\xi}_{c1} &= \frac{\partial}{\partial x}[A_{c1}(\alpha)x_{c1}]|_0\, \xi + \frac{\partial}{\partial w}[A_{c1}(\alpha)x_{c1}]|_0\, v \\
&\quad + \frac{\partial}{\partial u}[A_{c1}(\alpha)x_{c1}]|_0\, \eta + A_{c1}(\alpha)|_0\, \xi_{c1} \\
&\quad + \frac{\partial}{\partial x}[B_{c1}(\alpha)\dot{y}_1]|_0\, \xi + \frac{\partial}{\partial w}[B_{c1}(\alpha)\dot{y}_1]|_0\, v \\
&\quad + \frac{\partial}{\partial u}[B_{c1}(\alpha)\dot{y}_1]|_0\, \eta \\
&\quad + B_{c1}(\alpha)|_0 \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial h_1(x, w)}{\partial x}\bigg|_0 \xi + \frac{\partial h_1(x, w)}{\partial w}\bigg|_0 v \right] \\
&\quad + \frac{\partial}{\partial x}[B_{c2}(\alpha)(y_2 - r)]|_0\, \xi + \frac{\partial}{\partial w}[B_{c2}(y_2 - r)]|_0\, v \\
&\quad + \frac{\partial}{\partial u}[B_{c2}(\alpha)(y_2 - r)]|_0\, \eta + B_{c2}(\alpha)|_0(\theta_2 - \rho),
\end{aligned} \tag{24}$$

$$\begin{aligned}
\dot{\xi}_{c2} &= \frac{\partial}{\partial x}[C_{c1}(\alpha)x_{c1}]|_0\, \xi + \frac{\partial}{\partial w}[C_{c2}(\alpha)x_{c1}]|_0\, v \\
&\quad + \frac{\partial}{\partial u}[C_{c1}(\alpha)x_{c1}]|_0\, \eta + C_{c1}(\alpha)|_0\, \xi_{c1} \\
&\quad + \frac{\partial}{\partial x}[C_{c2}(\alpha)e]|_0\, \xi + \frac{\partial}{\partial w}[C_{c2}(\alpha)e]|_0\, v \\
&\quad + \frac{\partial}{\partial u}[C_{c2}(\alpha)e]|_0\, \eta + C_{c2}(\alpha)|_0(\theta_2 - \rho), \\
\eta &= \xi_{c2},
\end{aligned}$$

where $|_0$ means that the preceding expression is evaluated at the equilibrium point determined by $\alpha_0 \in \mathscr{A}$. Note that

$$\frac{\partial}{\partial x}[A_{c1}(\alpha)x_{c1}]|_0 = \sum_i x_{c1_{0i}} \left\{ \frac{\partial}{\partial x}[A_{c1_i}(\alpha)] \right\} \bigg|_0 = 0 \tag{25}$$

since $x_{c1_0} = 0$ (in (25) the subscript i represents both the ith element of x_{c1_0} and the ith column of A_{c1}). Similar results can be obtained for $(\partial/\partial w)[A_{c1}(\alpha)x_{c1}]|_0$, $(\partial/\partial u)[B_{c1}(\alpha)\dot{y}_1]|_0$ etc. Therefore the linearization of the controller has the form

$$\begin{aligned}
\dot{\xi}_{c1} &= A_{c1}(\alpha_0)\xi_{c1} + B_{c1}(\alpha_0)\dot{\theta}_1 + B_{c2}(\alpha_0)(\theta_2 - \rho), \\
\dot{\xi}_{c2} &= C_{c1}(\alpha_0)\xi_{c1} + C_{c2}(\alpha_0)(\theta_2 - \rho), \\
\eta &= \xi_{c2}.
\end{aligned} \tag{26}$$

It is easy to verify that the state matrix M of $\mathscr{T}_\ell(\mathscr{G}, \mathscr{C})(\alpha_0)$ is

$$M := \begin{bmatrix} A & 0 & B_2 \\ B_{c1}C_1A + B_{c2}C_2 & A_{c1} & B_{c1}C_1B_2 \\ C_{c2}C_2 & C_{c1} & 0 \end{bmatrix}, \tag{27}$$

where again we have omitted writing the explicit dependence of the elements of the matrix on the operating point. To complete the proof of the first part of the theorem, we now show that there exists a nonsingular matrix P such that $F = PMP^{-1}$. Earlier in the proof, it was shown that the matrix

$$\begin{bmatrix} A_{c1} & B_{c2} \\ C_{c1} & C_{c2} \end{bmatrix} \tag{28}$$

is invertible. Let

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} := \begin{bmatrix} A_{c1} & B_{c2} \\ C_{c1} & C_{c2} \end{bmatrix}^{-1}, \tag{29}$$

and set

$$P := \begin{bmatrix} I & 0 & 0 \\ -XB_{c1}C_1 & X & Y \\ -ZB_{c1}C_1 & Z & W \end{bmatrix}.$$

Routine algebra shows that

$$P^{-1} = \begin{bmatrix} I & 0 & 0 \\ B_{c1}C_1 & A_{c1} & B_{c2} \\ 0 & C_{c1} & C_{c2} \end{bmatrix}$$

and $F = PMP^{-1}$. Thus F and M have the same eigenvalues. In order to show that $T_{\mathscr{A}}(\mathscr{G}, \mathscr{C})(\alpha_0)(s) = T(\mathscr{G}_{\mathscr{A}}(\alpha_0), \mathscr{C}_{\mathscr{A}}(\alpha_0))(s)$, it suffices to prove that the linear controllers (8) and (26) have the same transfer functions from $(\theta_1', \theta_2', \rho')'$ to η. A simple computation shows that

$$\hat{\eta}(s) = C_{c1}(sI - A_{c1})^{-1}\left\{B_{c2}\frac{I}{s}[\hat{\theta}_2(s) - \hat{\rho}(s)] + B_{c1}\hat{\theta}_1(s)\right\}$$
$$+ C_{c2}\frac{I}{s}[\hat{\theta}_2(s) - \hat{\rho}(s)]$$

for both controllers, where $\hat{\eta}(s)$, $\hat{\theta}_1(s)$, $\hat{\theta}_2(s)$ and $\hat{\rho}(s)$ denote the Laplace transforms of η, θ_1, θ_2 and ρ, respectively.

Properties of $\mathscr{D}$-implementation. It is worth emphasizing the following important properties of the $\mathscr{D}$-implementation methodology.

1. The result in Theorem 4.1 holds for all the operating points of the plant in the set $\mathscr{E}$.
2. The structure of the gain-scheduled controller is easily obtained from that of the linear controllers.
3. Since all the closed-loop transfer functions of the local linearizations are preserved, at the level of local linear analysis, the method does not introduce any additional noise amplification, despite the presence of a differentiation operator.
4. At equilibrium, $x_{c2_0} = u_0$ and $x_{c1_0} = 0$. Therefore the input trimming value is naturally provided by the integrator block with state x_{c2_0}.
5. Suppose $D_{c2} = 0$. Then the integrators x_{c2} are directly at the input of the plant, which makes it straightforward to implement anti-wind-up schemes. This becomes necessary in applications where the input u is hard-limited owing to actuator saturation, for example.

We remark that in practical applications gain-scheduled controllers are often switched on after an initial phase where the plant is steered by an operator (manual mode) or by a linear time-invariant controller. It is then important to initialize properly the states of the gain-scheduled controller at the time of the switching in order to achieve bumpless transfer (see Hanus *et al.*, 1987; and references therein). Property 3 of the gain-scheduled controller (17) leads to a trivial solution to this problem when $D_{c2} = 0$. In fact, assume that the plant has been brought to an equilibrium point (x_0, w_0, u_0) at time $t = t_s \geq 0$ via manual control or by a fixed controller. Further assume that the inputs to the plant are the outputs of a chain of integrators with state $\tilde{\xi}$ (in the case of a fixed controller this means that the controller exhibits integral action). Then bumpless transfer is achieved by implementing (17) with x_{c2} replaced by $\tilde{\xi}$ and setting $x_{c1}(t_s) = 0$.

5. *Approximations of the $\mathscr{D}$ method*

The $\mathscr{D}$ method presented in Section 4 requires differentiating some of the plant's measured outputs. Except for the case where some of the derivatives have physical meaning and are available from dedicated sensors, this cannot be done in practice. In this section we indicate how to circumvent this difficulty when realizing controllers on analog computers. For the implementation of gain-scheduled controllers on digital computers, see Kaminer *et al.* (1994).

The main result of this section is stated in Theorem 5.1, where it is shown that the differentiation operator can be replaced by a causal system with transfer function $s/(\epsilon s + 1)$ (see Fig. 4), with $\epsilon > 0$, while having the linearization property recovered asymptotically as $\epsilon \to 0$. The result can be extended to the case where the differentiation operator is replaced by a strictly causal one.

Theorem 5.1. Let $\mathscr{G}$, $\mathscr{G}_{\mathscr{A}}(\alpha_0)$ and $\mathscr{C}_{\mathscr{A}}(\alpha_0)$ be as in Theorem 4.1. Consider the gain-scheduled controller

$$\mathscr{C}_\epsilon(\alpha) := \begin{cases} \dot{x}_{c1} = A_{c1}(\alpha)x_{c1} + \tilde{B}_{c1}(\alpha)(y_1 - \tilde{z}) + C_{c2}(\alpha)e, \\ \dot{x}_{c2} = C_{c1}(\alpha)x_{c1} + \tilde{D}_{c1}(\alpha)(y_1 - \tilde{z}) + D_{c2}(\alpha)e, \\ \epsilon\dot{\tilde{z}} = -\tilde{z} + y_1 \\ e = y_2 - r, \\ u = x_{c2} + D_{c2}(\alpha)e, \\ \alpha = v(w, u, y), \end{cases} \qquad (30)$$

where α is the scheduling variable, ϵ is a positive real parameter, $\tilde{B}_{c1}(\alpha) = B_{c1}(\alpha)/\epsilon$, $\tilde{D}_{c1}(\alpha) = D_{c1}(\alpha)/\epsilon$ and $\tilde{z} \in \mathbb{R}^r$ denote the fast states of the controller, with $\dim(\tilde{z}) = \dim(y_1) = m$.

Then as $\epsilon \to 0$, the following hold.

1. m of the eigenvalues of $\mathscr{T}_{\mathscr{A}}(\mathscr{G}, \mathscr{C}_\epsilon)(\alpha_0)$ approach infinity, while the rest approach the eigenvalues of $\mathscr{T}(\mathscr{G}_{\mathscr{A}}(\alpha_0), \mathscr{C}_{\mathscr{A}}(\alpha_0))$.
2. $\|\mathscr{T}_{\mathscr{A}}(\mathscr{G}, \mathscr{C}_\epsilon)(\alpha_0) - \mathscr{T}(\mathscr{G}_{\mathscr{A}}(\alpha_0), \mathscr{C}_{\mathscr{A}}(\alpha_0))\|_{p,i}$ tends to zero for every $1 \leq p \leq \infty$, where $\|.\|_{p,i}$ denotes the $\mathscr{L}_p$-induced (input-output) operator norm. In particular, when $p = 2$, it follows that the transfer matrix $T_{\mathscr{A}}(\mathscr{G}, \mathscr{C}_\epsilon)(\alpha_0)(s)$ approaches $T(\mathscr{G}_{\mathscr{A}}(\alpha_0), \mathscr{C}_{\mathscr{A}}(\alpha_0))(s)$ in the H_∞ norm, that is, uniformly over the right-half complex plane.

The main ingredients of the proof come from the theory of single-parameter singularly perturbed systems (Kokotović *et al.*, 1986) and from the work of Pascoal *et al.* (1991) and Vidyasagar (1984) on robust stabilizability of those systems. For details of the proof the reader is referred to Kaminer *et al.* (1994).

At this point, we revisit the example of Section 4.1 with d set equal to zero. The corresponding (singularly perturbed) gain-scheduled controller is given by

$$\mathscr{C}_\epsilon(x) := \begin{cases} \dot{x}_c = -x + r - \dfrac{(\beta - 2x)(x - \tilde{z})}{\epsilon}, \\ \dot{\tilde{z}} = \dfrac{-\tilde{z} + x}{\epsilon}, \\ u = x_c. \end{cases} \qquad (31)$$

The feedback system consisting of the plant (10) and

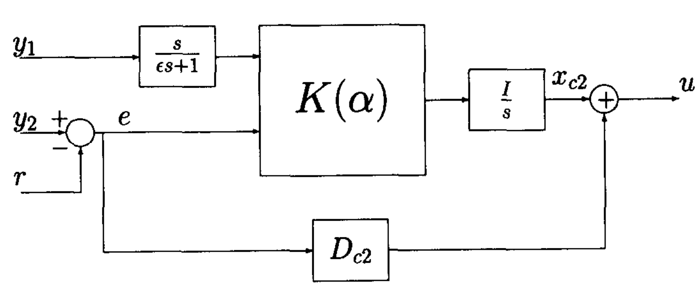


Fig. 4. Analog implementation of $\mathscr{C}_{\mathscr{A}}(\alpha)$.

controller (31) can be linearized about the equilibrium point $(x_0, r_0 = x_0, x_{c0} = x_0^2, \tilde{z}_0 = x_0)$, leading to the system

$$\mathcal{T}_{\mathcal{A}}(\mathcal{G}, \mathcal{C}_\epsilon)(x_0) := \begin{cases} \dot{\xi} = -2x_0\xi + \xi_c, \\ \dot{\xi}_c = \left(-1 - \dfrac{\beta - 2x_0}{\epsilon}\right)\xi + \dfrac{\beta - 2x_0}{\epsilon}\tilde{\vartheta} + \rho, \\ \dot{\tilde{\vartheta}} = \dfrac{-\tilde{\vartheta} + \xi}{\epsilon}. \end{cases} \tag{32}$$

It is easy to check that the characteristic polynomial associated with (32) is given by $(\epsilon(s^3 + 2x_0s^2 + s) + (s^2 + \beta s + 1)$. A root-locus type of argument shows that as $\epsilon \to 0$, two of the eigenvaues of (32) tend to the roots of $s^2 + \beta s + 1$, while the third tends to infinity along the negative real axis. Furthermore,

$$\mathscr{E}(s) = \frac{\hat{\xi}(s)}{\hat{\rho}(s)} - \frac{1}{s^2 + \beta s + 1}$$
$$= \frac{\epsilon s^2(\beta - 2x_0)}{(s^2 + \beta s + 1)[\epsilon s^2 + s^2(1 + 2\epsilon x_0) + s(\beta + \epsilon) + 1]}.$$

It is now straightforward to show, using the results of Kokotović *et al.* (1986, p. 57) on the asymptotic behaviour of the closed-loop eigenvalues that

$$\lim_{\epsilon \to 0} \|\hat{\mathscr{E}}(.)\|_\infty = \lim_{\epsilon \to 0} \sup\{|\hat{\mathscr{E}}(j\omega)| : \omega \in \mathbb{R}\} = 0.$$

6. *Conclusions and suggestions for future work*

A new method has been described to implement gain-scheduled controllers for nonlinear plants. The starting point is a family of linear controllers with integral action, designed for linearizations of a nonlinear plant about constant operating points. Based on that family, the method produces a gain scheduled controller that preserves the closed-loop eigenvalues as well as the input–output properties of the original linear closed-loop systems locally, about each operating point. The method is simple to apply, and leads to a nonlinear controller with a structure similar to that of the original controllers. Future work will concentrate on the following issues: (i) determining the extent of asymptotic stability of the nonlinear gain scheduled system; (ii) studying the impact of the dynamics involved in computing the scheduling variable (from the observed inputs and outputs to the plant) on the system's performance.

Acknowledgements—This work was supported in part by a Research Initiation Grant of the Naval Postgraduate School, NSF under Grant ECS-9096109, AFOSR under Grant F49620-93-1-0246 and ARO under Grant DAAH04-93-G-0012. A. M. Pascoal was supported by the Commission of the European Communities under Contract MAS2-CT92-0021, and by JNICT under Contract PMCT/C/TIT/702/90.

References

Åström, K. J. and B. Wittenmark (1988). *Adaptive Control.* Addison-Wesley, Reading, MA.

Baumann, W. T. (1988). Feedback control of multiinput nonlinear systems by extended linearization. *IEEE Trans. Autom. Control,* **AC-33,** 193–197.

Baumann, W. T. and W. J. Rugh (1986). Feedback control of nonlinear systems by extended linearization. *IEEE Trans. Autom. Control,* **AC-31,** 40–46.

Baumann, W. T. and W. J. Rugh (1987). Feedback control of analytic nonlinear systems by extended linearization. *SIAM J. Control Optim.,* **25,** 1341–52.

Hanus, R., M. Kinnaert and J. Henrotte (1987). Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica,* **23,** 729–739.

Isermann, R. (1981). *Digital Control Systems.* Springer-Verlag, Berlin.

Kaminer, I., P. P. Khargonekar and G. Robel (1990). Design of localizer capture and track modes for a lateral autopilot using $\mathcal{H}_\infty$ synthesis. *IEEE Control Syst. Mag.,* **10**(4), 13–21.

Kaminer, I., A. M. Pascoal, C. Silvestre and P. P. Khargonekar (1991). Design of a control system for an underwater vehicle using $\mathcal{H}_\infty$ systhesis. In *Proc. 30th IEEE Conf. on Decision and Control,* Brighton, U.K., pp. 2350–2355.

Kaminer, I., A. M. Pascoal, P. P. Khargonekar and C. Thompson (1993). A velocity algorithm for the implementation of gain-scheduled controllers. In *Proc.* 1993 *European Control Conf.,* Eindhoven, The Netherlands, pp. 787–792.

Kaminer, I., A. M. Pascoal, P. P. Khargonekar and E. Coleman (1994). A velocity algorithm for the implementation of gain-scheduled controllers. Internal Report, Naval Postgraduate School.

Kokotović, P., H. Khalil and O'Reilly (1986). *Singular Perturbation Methods in Control: Analysis and Design.* Academic Press, London.

Lawrence, D. A. and W. J. Rugh (1993). Gain scheduling dynamic linear controllers for a nonlinear plant. In *Proc. 32nd IEEE Conf. on Decision and Control,* San Antonio, TX, pp. 1024–1029.

Pascoal, A., P. P. Khargonekar and R. Ravi (1991). Weighted robust stabilization of linear time-varying singularly perturbed systems (to be submitted). For an abridged version, see *Proc.* 1991 *European Control Conf.,* Grenoble, pp. 1706–1711.

Rugh, W. J. (1990). Analytical framework for gain scheduling. *IEEE Control Syst. Mag.,* **11**(1), 74–84.

Rugh, W. J. (1991). Input–output pseudo linearization for nonlinear systems. In *Proc. 30th IEEE Conf. on Decision and Control,* Brighton, U.K., pp. 74–84.

Salo, P. (1987). Delta coordinate controller implementation to avoid the 'fictitious gain' problem arising from gain-scheduling. In *Proc. Boeing Intercompany Controls Symp.,* Seattle, WA, pp. 200–205.

Shamma, J. S. (1988). Analysis of gain scheduled control for nonlinear plants. PhD thesis, MIT.

Shamma, J. S. and M. A. Athans (1990). Analysis and design of gain scheduled control systems. *IEEE Trans. Autom. Control,* **AC-35,** 898–907.

Slotine, J. J. and W. Li (1991). *Applied Nonlinear Control.* Prentice-Hall, Englewood Cliffs, NJ.

Stein, G., G. Hartmann and R. Hendrick (1977). Adaptive control laws for F-8 flight test. *IEEE Trans. Autom. Control,* **AC-22,** 758–767.

Vidyasagar, M. (1984). The graph metric for unstable plants and robustness estimates for feedback stability. *IEEE Trans. Autom. Control,* **AC-29,** 403–417.

Wang, J. and W. J. Rugh (1987). Feedback linearization families for nonlinear systems. *IEEE Trans. Autom. Control,* **AC-32,** 935–940.